(Tensor) Triangulated Categories

What are they good for?

David Rubinstein

Why Care about these categories?

The axioms suck (not really, but still)

The definition/axioms of triangulated categories are notoriously viewed as being bad. In fact, many people are not convinced that the axioms as they currently exist will remain in their current form.
So, rather jumping right into the axioms, let us first see that they at least provide a wide range of examples. (The range of applications provides a good argument for the side of the axioms it should be noted)

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7. C*- Algebras in KK-Theory

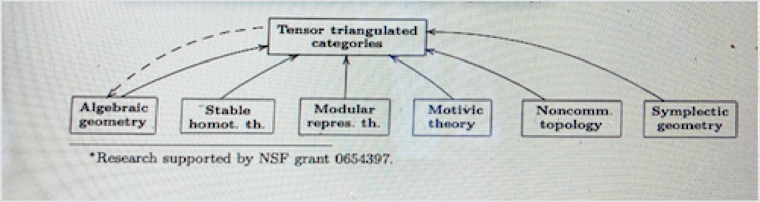

Figure: Picture stolen from Paul Balmer- depicts the broad scope of TT-Categories

A Framework to Keep in Mind

Groups VS Commutative Rings

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4. Next talk will be analogues to Ring Theory- via the work of Balmer et, al one can push this analogy quite far through the use of the Triangulated Spectrum.

Some Leading Questions

Unification

1. Since so many examples of these Triangulated Categories exist ranging from algebra/geometry (examples 2,3,4) to topology (1,5,6) to analysis (7) the vague hope is that by studying "Triangulated Categories" writ large, one can learn things about all these topics in one fell swoop.

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2. Moreover, one can ask if we can transfer ideas from one branch to another using this framework of Triangulated Categories. Beren likes to describe the image above as being a conveyor belts of sorts- you feed in data from one example, and use the conveyor belt of TT-Categories to spit out that data in another example.

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3. For example- Can we relate the ideas of line bundles in Algebraic-Geometry and Endotrivial Modules in Representation Theory? Moreover, can we relate Spec(R) for a commutative ring with Support Varieties $\mathcal{V}_G := \mathrm{Proj}(H^*(G, k))$

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4. There is a famous nilpotence theorem of Hopkins and Smith in Algebraic Topology- are there analogous nilpotent theorems in Other contexts? In Alg Topology this nilpotence theorem provides a stratification for our category, can we expect the same for other nilpotence theorems?

The Axioms

Oh Lord the Axioms

A *pre-triangulated category* is an additive category $\mathcal{T}$ equipped with an auto-equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ (called *suspension*) and a class of diagrams (called *distinguished triangles*) $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ satisfying the axioms:

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1. *(T0- Bookkeeping axiom)* The distinguished triangles are closed under isomorphisms and rotations; That is, a triangle is distinguished iff $b \xrightarrow{g} c \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma f} \Sigma b$ is distinguied. Furthermore, For all $x \in \mathcal{T}$, the triangle $x \xrightarrow{\mathrm{Id}} x \longrightarrow 0 \longrightarrow \Sigma x$ is distinguished;

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3. *(T2- Morphism Axiom)* Given the diagram whose rows are distinguished triangles and whose left square commutes

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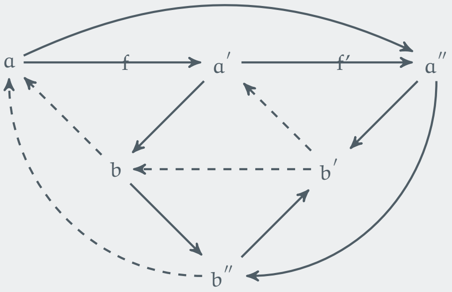
there exists $\gamma : c \to c'$ such that the entire diagram commutes.

The dreaded Octahedral Axiom

A *Triangulated Category* is a pre-Triangulated Category that satisfies the further axiom

4. *(T3)- The Octahedral Axiom* Any two morphisms $a \xrightarrow{f} a' \xrightarrow{f'} a''$ fit into a diagram



where the dashed lines represent maps into the suspension and where the four triangles inside are distinguished.

Motivation 1 for Axioms/Terms

CW-Complexes

1. In a beginning Algebraic Topology class, one studies so called CW-Complexes. Now given any morphism of $X \xrightarrow{f} Y$ of CW-complexes one can form another CW-Complex, called the **"Cone of f"**, and denoted C(f), that comes with a map $Y \to C(f)$.

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3. Finally, for any map $X \xrightarrow{f} Y$ of CW-complexes there is a natural map $C(f) \xrightarrow{h} \Sigma X$.
4. Putting this all together, for any map of CW- complexes, we get morphisms $X \xrightarrow{f} Y \xrightarrow{g} C(f) \xrightarrow{h} \Sigma X$.

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Complexes of R-Modules

1. Recall that the Category of Chain Complexes of R-Modules has objects

$$A = \cdots \to A_n \stackrel{d_n}{\to} A_{n-1} \stackrel{d_{n-1}}{\to} A_{n-2} \to \cdots$$

with A_i an R-Module and where the d_i are R-Module Homomorphisms called the differentials.

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5. Then putting it all together we get a sequence of morphisms of chain complexes

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} \Sigma A$$

   where i and p are the canonical inclusion and projection maps.

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5. If $a_1 \bigoplus a_2 \xrightarrow{f} b_1 \bigoplus b_2 \xrightarrow{g} c_1 \bigoplus c_2 \xrightarrow{h} \Sigma(a_1 \bigoplus a_2)$ is a triangle, so are $a_i \xrightarrow{f_i} b_i \xrightarrow{g_i} c_i \xrightarrow{h_i} \Sigma a_i$

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1. "The Five Lemma of Triangles"- Suppose we are given a morphism of triangles

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2. One might imagine that given a map $a \xrightarrow{f} b$ we could complete it into 2 different triangles. The above result actually shows such a triangle is unique up to (a non-unique) isomorphism. (take $a = a', b = b', \alpha = \beta = 1$)

Basic Applications

Last bit on Intuition

1. "Split Triangles". Suppose $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$ is a triangle such that there exists a retraction of g. Then there exists an isomorphism of triangles

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3. Pullbacks and Pushouts exist. (up to a weak version- the maps out of/into them are unique but up to a non-unique isomorphism). They are often called **Homotopy Cartesian Squares**. Moreover, the completion of the triangle guaranteed in Axiom T2 can be chosen in such a way that makes the middle square Homotopy Cartesian.

On the Road to Localizations

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2. Now is a good time to mention- no results above used the Octahedral Axiom. In fact, Neeman doesn't even give the 4th Axiom in his textbook on Triangulated Categories until after the sections covering the material above. His 4th axiom is actually different than the Octahedral Axiom(his involves so called mapping cones) but he eventually proves they are equivalent.
3. We will need it for the slides to come however.

Towards Thick Subcategories

Killing Cones

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2. We let $A_{\mathcal{S}}$ be the sub collection of morphisms of $\mathcal{T}$ be given by all morphisms whose cone lies in $\mathcal{S}$. That is, $f \in A_{\mathcal{S}}$ iff there is some x,y such that $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ is a triangle with $z \in \mathcal{S}$. Then note,

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2. We let $\mathbf{A}_{\mathcal{S}}$ be the sub collection of morphisms of $\mathcal{T}$ be given by all morphisms whose cone lies in $\mathcal{S}$. That is, $f \in A_{\mathcal{S}}$ iff there is some x,y such that $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ is a triangle with $z \in \mathcal{S}$. Then note,
   - 2.1 Every isomorphism f is in $A_{\mathcal{S}}$.
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3. **A triangulated subcategory $\mathcal{D}$ is called Thick** if it is closed under direct summands.
   3.1 One should think of thick subcategories as being the analogue of normal subgroups. In group Theory, we have a correspondence between normal subgroups and kernals of group homomorphisms. Is something like that true here?

Triangulated Functors and Kernals

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   - 5.2 Is the converse true? Are all thick subcategories kernals of a triangulated functor?

Verdier Localization

Main Thrm

Let $\mathcal{D}$ be a thick subcategory of a triangulated category $\mathcal{T}$. Then there exists a triangulated category, denoted $\mathcal{T}/\mathcal{D}$ and a universal triangulated functor $\mathrm{Q} : \mathcal{T} \to \mathcal{T}/\mathcal{D}$ such that $\ker(\mathrm{Q}) = \mathcal{D}$. This pair is universal in the sense that if there is a triangulated category $\mathcal{F}$ and functor $\mathrm{G} : \mathcal{T} \to \mathcal{F}$ with $\mathcal{D} \subseteq \ker(\mathrm{G})$, then G factors uniquely as $\mathcal{T} \xrightarrow{\mathrm{Q}} \mathcal{T}/\mathcal{D} \xrightarrow{\overline{\mathrm{G}}} \mathcal{F}$. Indeed

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3. Since we are inverting morphisms whose cone lands in $\mathcal{D}$, one can show that every object of $\mathcal{D}$ is isomorphic to 0 in $\mathcal{T}/\mathcal{D}$.
4. The process is very gross and one loses control over the morphisms in the localized category pretty easily. In fact, the collection of morphisms will in general be a class (and not a set) in the localized category. We can get around this in some sneaky ways however- hopefully we can discuss this soon.
5. Side note- The way in which one formally inverts morphisms is just a massive generalization of how one localizes a ring.

Derived Category of a Ring

An example of the above Thrm

Let us show this powerful theorem in action. Let R be a ring and let $\mathcal{T} = \mathrm{K}(\text{R-Mod})$ denote the homotopy category of chain complexes of R-modules. This is a classic example of a triangulated category. Now consider the subcategory of so called **Acyclic** objects, defined as $Acy(\mathcal{T}) = \{A \in \mathcal{T} : H^i(A) = 0$ for all $i\}$. That is, the collection of all chain complexes with trivial Homology. Then

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4. Much of the historical motivation of the work above was to find a category in which these quasi-isomorphisms actually are isomorphisms- the above does just that.

Variations of a Theme

Similar examples

1. Often, we are focused only on chain complexes that are either bounded above, below, or just plain old bounded. The collection of all those form triangulated categories in their own right, denoted as $K^{+}(R-\mathrm{Mod}), K^{-}(R-\mathrm{Mod}), K^{b}(R-\mathrm{Mod})$. respectively.

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2. One then looks at the acyclic objects in each of these categories and again forms the Verdier quotient for all of these.
3. Alternatively, recall that D(R) has the same objects as $K(R-\mathrm{Mod})$, so we could talk about bounded (above/below) complexes in D(R)- again denoted $D^{+}(R), D^{-}(R), D^{b}(R)$.

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4. It is a non trivial fact that these are the verdier quotients of the above. That is
   4.1 $D^+(R) \cong K^+(R - \mathrm{Mod})/\mathrm{Acy}(K^+(R - \mathrm{Mod}))$
   4.2 $D^-(R) \cong K^-(R - \mathrm{Mod})/\mathrm{Acy}(K^-(R - \mathrm{Mod}))$
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Quick Applications of Derived Category

Ext Groups

1. Recall that one of the major tools in group cohomology theory are the so called Ext Groups. These are the Right Derived Functors of Hom and one goes through a lot of work showing that they "fix" the non-exactness of Hom
(that is, for any pair of morphisms $A \to B \to C$, and any R-Mod X, there is a LES $0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to \mathrm{Ext}^1(X, A) \to \cdots$)

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4. YUP!!! Moreover, recall that $\mathrm{Hom}_{D(R)}(X, -), \mathrm{Hom}_{D(R)}(-, Y)$ are Homological/Cohomological, so they automatically turn any triangle into a LES. In particular, if $0 \to A \to A' \to A'' \to 0$ is a SES, then this gives an exact triangle in $D(R)$. So one recovers the LES expected from Ext by applying $\mathrm{Hom}_{D(R)}(\Sigma^{-1}X, -)$ to this triangle. Moreover, this shows one can construct these sequences even without enough injective/projectives! (A similar Statement is true for Tor as well)

An Eye towards Bausfield Localization 1

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3. As a side note - Interestingly enough, one would probably think the relationship of stab(kG) to Stab(kG) should be the same as the relationship of $D^b(R)$ to $D(R)$. This is actually incorrect in a precise sense- and hints at a possible "failure" for D(R).

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4. Now on the face of it, this seems disjoint from the previous topic. However, it is a theorem of Rickard (1989) that one can realize stab(kG) as a Verdier Quotient of a derived Category, namely

$$stab(kG) \cong D^b(kG)/K^b(P-Mod).$$

   where **P-Mod is the full subcategory of projective modules.** In fact this is true for any self injective algebra.

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5. This is the type of Localization one does in SH. It was Bausfield who realized you could do this game for any Homology Functor out of SH (that is, for any Homology functor with kernal S, there is a Localization functor with that kernal too)

Tensor Structure

What about the Tensor

1. All of the above has been about so called Triangulated Categories- and one can get pretty far working just in that world.
2. However, most of the examples "in nature," and indeed all of the examples given in this talk have an added "tensor product" structure on them.
3. It is when this tensor structure is added to the mix that things get really interesting- and including the tensor opens the door to so called "Tensor Triangulated Geometry", and with that a world of unification.