Homework 10 Answers
David Rubinstein - Math 100 - Fall 2019

12.1-8) Consider the set f=$\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is f a function?

Solution: We have that f is not a function since there does not exist a y such that (2,y) $\in f$ (meaning 3y=2 has no integer solutions).

12.2-4) A function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined by f(z)=(2z, z+3). Determine whether or not f is injective or surjective.

Solution: We have f is clearly not surjective since no odd number is hit in the first component.
Now assume f(z)=f(z'). Then z+3=z'+3 so z=z'. Hence f is injective.

12.2-16) This question concerns functions $f : \{A, B, C, D, E\} \to \{1, 2, 3, 4, 5, 6, 7\}$. How many functions are there? How many injections are there? How many surjections are there? How many bijections are there?

Solution: First we have that the size of the domain is smaller than the size of the codomain. Thus we can conclude there are no surjections, and hence no bijections either.
Now we count how many functions there are. For each point we have 7 choices to choose where to map to. Thus, since we have 5 points, we have 7^5 total functions.
Now we count how many injections there are. We can send A to any 7 points; then we can send B to any remaining 6 points; C to any remaining 5 points, and so on. Thus we get $7 \times 6 \times 5 \times 4 \times 3 = 2520$ total injections.

12.4-8) Consider the functions $f, g : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ by f(m,n)=(3m-4n, 2m+n), and g(m,n)=(5m+n,m). Find formulas for both $f \circ g, g \circ f$.

Solution: We have $(f \circ g)(m,n) = f(g(m,n)) = f(5m+n,m) = (3(5m+n) - 4m, 2(5m+n) + m) = (11m + 3n, 11m + 2n)$
Now $(g \circ f)(m,n) = g(f(m,n)) = g(3m - 4n, 2m + n) = (5(3m - 4n) + 2m + n, 3m - 4n) = (17m - 19n, 3m - 4n)$

(12.5-2) We have that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ given by $f(x) = \dfrac{5x+1}{x-2}$ is a bijection. Find the inverse of f.

Solution: We let $x = \dfrac{5y+1}{y-2}$ and solve for y. We get that $y = \dfrac{-2x-1}{5-x}$ is the inverse.

(12.5-8) Is the function $\theta : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ given by $\theta(X) = \overline{X}$ bijective? If so find the inverse.

Solution: This map is a bijection. To see this, first assume $\theta(X) = \theta(Y)$. Then we have $\overline{X} = \overline{Y}$ so $x \notin X \iff x \notin Y$, ie $x \in X \iff x \in Y$ showing that X=Y. Hence θ is injective. Now let $A \subseteq \mathbb{Z}$. If A was all of $\mathbb{Z}$ then we have $\theta(\phi) = A$ and if $A = \phi$ then $\theta(\mathbb{Z}) = A$ so assume that A is properly contained in $\mathbb{Z}$. Then let $B = \mathbb{Z} - A$. Then note that $\theta(B) = A$. Thus θ is surjective.
In proving that θ was surjective we really showed what the inverse of θ is: namely itself!!. That is $\theta \circ \theta = id_{\mathcal{P}(\mathbb{Z})}$
Fun unimportant fact: this shows that the subgroup of the group that I defined in my problems this last week $Invert(\mathcal{P}(\mathbb{Z}))$ generated by θ is "cylic of order 2". This really means that one can get an "isomorphism" between this subgroup and the integers modulo 2, $\mathbb{Z}_2$! This follows from an easy to prove, but important fact you will learn in an intro group theory class; namely every finite cyclic group is "equal" (ie, isomorphic) to $\mathbb{Z}_n$ for some n. This gives some justification for why we care about integers modulo n.

(12.6-6) Given a function $f : A \to B$ and a subset $Y \subseteq B$ is it always true that $f(f^{-1}(Y)) = Y$?

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Then let $Y = \{-4, -1, 0, 1, 4\}$. Then we have that $f^{-1}(Y) = \{0, 1, 2\}$. Thus we get $f(f^{-1}(Y)) = f(\{0.1, 2\} = \{0, 1, 4\} \subset Y$, so we conclude it is false.

(12.6-10) Given $f : A \to B$ and subsets $Y, Z \subseteq B$ prove that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$

Solution: Recall that $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$.
Thus

$$\begin{aligned} f^{-1}(Y \cap Z) &= \{x \in A : f(x) \in Y \cap Z\} \\ &= \{x \in A : f(x) \in Y \wedge f(x) \in Z\} \\ &= \{x \in A : x \in f^{-1}(Y) \wedge f^{-1}(Z)\} \\ &= \{x \in A : x \in f^{-1}(Y) \cap f^{-1}(Z)\} \\ &= f^{-1}(Y) \cap f^{-1}(Z) \end{aligned}$$