Homework 5 Answers
David Rubinstein - Math 100 - Fall 2019

3.9-2) You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

Solution: We get that there are 4 suits in the deck, so we can think about placing each card into the box representing its corresponding suit. We have from the generalized pigeonhole principle that, when we choose n cards at least one of the boxes has at leas $\lceil \frac{n}{4} \rceil$ cards. Thus we want $\lceil \frac{n}{4} \rceil \geq 5$. This is true iff $4 < \frac{n}{4} < 5$. Hence, $n > 20$ so the smallest such n is 21.

(3.9-6) Given a sphere S, a great circle of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A hemisphere is the union of the great circle and one of these two parts. Show that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

Solution: This requires a bit of geometry that we need to remember. We use the following fact: that in $\mathbb{R}^3$ any 3 points uniquely determine a plane. Now we proceed as follows: place the 5 points arbitrarily on the sphere; now take any two of the points, and the origin of the sphere and draw the plane through those three points. Where this plane intersects the sphere gives us a great circle containing two of our 5 points. Now the sphere has been cut in 2, and there are 3 points remaining so by the pigeonhole principle one half of the sphere must contain at least 2 points. Hence that half unioned the great circle is our hemisphere with at least 4 points.

(3.10-4) Use combinatorial proof to show that P(n,k)= P(n-1,k) + kP(n-1,k-1)

Solution: We can think of the LHS of this equations as counting the number of different ways we can give k jobs in a group of n people. We show the RHS is the same as follows: We can either give person n a job or not. If we do not give them a job, then we still have k jobs to fill out of n-1 people, and the number of different ways of doing that is P(n-1,k). Now if we give person n a job, we have k different jobs to assign them to. Then we have k-1 jobs to fill out of n-1 people, and the number of different ways of doing this is (by the multiplicative principle) kP(n-1,k-1). Hence adding up those two options (person n has a job or not) we get what we desire.

(4-6) Suppose a,b,c $\in \mathbb{Z}$ Show that if $a|b$ and $a|c$ then $a|(b+c)$

Solution: Since $a|b$ we have that $ax_1 = b$ for $x_1 \in \mathbb{Z}$. Simillarly since $a|c$ we have that $ax_2 = c$ for $x_2 \in \mathbb{Z}$ Hence $b + c = ax_1 + ax_2 = a(x_1 + x_2)$ where $x_1 + x_2 \in \mathbb{Z}$. This shows that $a|(b + c)$ as desired.

(4-8) Suppose $a \in \mathbb{Z}$. Show that if $5|2a$ then $5|a$

Solution: We have by assumption that $5c = 2a$ for $c \in \mathbb{Z}$. Note the RHS is even, and so the LHS must be even as well. Now since 5 is odd, we must have that c is even (or else 5c would be odd). Hence c=2k for some $k \in \mathbb{Z}$. Thus 5(2k)=2a, so we can cancel out the 2 to get a=5k, showing that $5|a$. (Note, we needed to show that c was even to guarantee that when we divided by 2, we got an integer back.)
An unimportant remark: this is something special about prime numbers in fact: we will show that an equivalent definition of being prime is the following: a number p is prime iff whenever $p|ab$ then $p|a$ or $p|b$. It is this definition that generalizes nicer as you shall see if you eventually take any abstract algebra courses (for example, prime ideals in ring theory).

(4-10) Suppose $a|b$. Show that $a|(3b^3 - b^2 + 5b)$

Solution: We have, by assumption, that $b = ak$ for $k \in \mathbb{Z}$.
Hence $3b^3 - b^2 + 5b = 3(ak)^3 - (ak)^2 + 5(ak) = a(3a^2k^3 - ak^2 + 5k)$ where $3a^2k^3 - ak^2 + 5k \in \mathbb{Z}$

(4-14) If $n \in \mathbb{Z}$ then $5n^2 + 3n + 7$ is odd.

Solution: We need to do this in cases. We first consider when n is odd, and then when n is even.
i) Assume n=2k+1 for $k \in \mathbb{Z}$. Then $5n^2 + 3n + 7 = 5(2k + 1)^2 + 3(2k + 1) + 7 = 5(4k^2 + 4k + 1) + 6k + 10 = 2(10k^2 + 13k + 7) + 1$ so odd.
ii) Now assume n=2c for $c \in \mathbb{Z}$. Then $5n^2 + 3n + 7 = 5(2c)^2 + 3(2c) + 7 = 2(10k^2 + 3c + 3) + 1$ so again odd.

(4-24) Show that if $n \in \mathbb{N}, n \geq 2$ then n!+2, n!+3, … n! +n are all composite.

Solution: We have that $n! = n(n-1)\cdots(3)(2)(1)$ so n!+2 is divisible by 2, n!+3 is divisible by 3, and so on up until n!+n being divisible by n. Hence they are all composite.

(5-6) Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$

Solution: We show that if $x \leq -1$ then $x^3 - x \leq 0$. Therefore, assume that $x \leq -1$. Then we get that $x^3 \leq -1$ (be careful with inequalities here, if it was just squared we would get $x^2 \geq 1$). Thus we have $x^3 - x \leq -1 - (-1) = 0$

(5-18) Use either direct or contrapositive proof to show that, for $a, b \in \mathbb{Z}$, $(a+b)^3 \equiv a^3 + b^3 \pmod 3$

Solution: I'll prove this by "brute force". We have $(a+b)^3 = (a+b)(a^2+2ab+b^2) = a^3+3a^2b+3ab^2+b^3 \equiv a^3+b^3 \pmod 3$. There are a lot of more clever, cute ways of proving this in fact. One could use the binomial theorem and note that 3 divides all terms but the first and last term, giving us the result we wanted. More advanced proofs require the language of group theory, see below.
In fact, this is a great result (that is often called the "freshman's dream"- literally there is a Wikipedia page about this lol) about prime numbers and so called "finite fields" that you will also study if you continue in abstract algebra. One can show that this result holds for any prime number; and since we have some gigantic prime numbers, we certainly need a more efficient way of proving this in general. (Look up the frobenius automorophism if you are curious about this)

(Question 4 of my practice midterm)
i) If a is an odd integer then $a^2 + 3a + 5$ is odd

Solution: We have that $a = 2k+1$ for some $k \in \mathbb{Z}$. Thus $a^2+3a+5 = (2k+1)^{+}3(2k+1)+5 = 2(2k^2+4k+4)+1$ where $2k^2+4k+4 \in \mathbb{Z}$. Hence it is odd.

ii) If two numbers are of opposite parity then their product is even.

Solution: Without loss of generality (WLOG) assume that $n = 2k$ for $k \in \mathbb{Z}$ and $m = 2c+1$ for c $\in \mathbb{Z}$. Then nm=(2k)(2c+1)=2(2kc+k) where $2kc+k \in \mathbb{Z}$. Hence nm is even.

(iii) If $n \in \mathbb{Z}$ then $n^2 + 3n + 4$ is even.

Solution: We consider cases.
i)n=2k for $k \in \mathbb{Z}$ Then $n^2 + 3n + 4 = 4k^2 + 6k + 4 = 2(2k^2 + 3k + 2)$ where $2k^2 + 3k + 2 \in \mathbb{Z}$
ii) n=2c+1 for $c \in \mathbb{Z}$. Then $n^2 + 3n + 4 = 4c^2 + 4c + 1 + 6c + 3 + 4 = 2(2c^2 + 5c + 4)$ where we have $2c^2 + 5c + \in \mathbb{Z}$

(iv) Suppose a,b $\in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are both odd.

Solution: We will prove the contrapositive. We assume a or b is even and show that $a^2(b^2 - 2b)$ is even.
i) Assume that a=2k for k $\in \mathbb{Z}$. Then $a^2(b^2 - 2b) = 2(2k^2)(b^2 - 2b)$ is always even.
ii) Now assume that b=2c is even. I'll show $a^2(b^2 - 2b)$ is even a little differently to change things up. We have proved in class (I believe, if not prove it to yourself that) that an even number times any number is even and that an even number plus (or minus) an even number is even. Hence b^2 and $2b$ are even and thus $b^2 - 2b$ is even. Thus $a^2(b^2 - 2b)$ is even since $b^2 - 2b$ is even.

(v) Let a,b $\in \mathbb{Z}, n \in \mathbb{N}$. Show that if $a \equiv b$ (mod n) then $a^3 \equiv b^3$ (mod n).

Solution: Since $a \equiv b$ (mod n) we have $n|(a - b)$, ie $nx_1 = a - b$ for some $x_1 \in \mathbb{Z}$. Now $a^3 - b^3 = (a - b)(a^2 + 2ab + b^2)$. Hence plugging in we get $a^3 - b^3 = nx_1(a^2 + 2ab + b^2)$ so $n|a^3 - b^3$ showing that $a^3 \equiv b^3$ (mod n).