Homework 6 Answers
David Rubinstein - Math 100 - Fall 2019

5-8) Suppose $x^5 - 4x^4 + 4x^3 - x^2 + 3x - 4 \geq 0$ then $x \geq 0$

Solution: Assume $x \leq 0$. Then it is easy to see that $x^5, -4x^4, 4x^3, 3x$ are all negative. Thus adding the whole expression (and being much more careful and precise than I am right now) we get that $x^5 - 4x^4 + 4x^3 - x^2 + 3x - 4 \leq 0$

5-14) If a,b $\in \mathbb{Z}$ and a,b have same parity then 3a+7 and 7b-4 do not.

Solution: First suppose a and b are even. Then 3a is even so 3a+7 is odd. Now 7b is even so 7b-4 is even. Thus they have opposite parity.
Now suppose a and b are both odd. Then 3a is also odd, so 3a+7 is even. Similarly 7b is odd so 7b -4 is odd. Thus again they have opposite parity.

6-6) If a,b $\in \mathbb{Z}$ then $a^2 - 4b - 2 \neq 0$

Solution: Suppose for the sake of contradiction that $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2$ so $a = \pm\sqrt{4b+2}$. Now since a is an integer, we must have that 4b+2 is a perfect square. Yet 4b+2 $\equiv$ 2 mod 4 and we proved on my Section 6 Question 1b that an integer cannot be a perfect square if it is congruent to 2 mod 4. This means a is not an integer, a contradiction.

(6-10) There exist no integers a and b such that 21a +30b=1

Solution: Assume there exist integers a and b such that 21a+30b=1. Now we can factor out a 3 to get 3(7a+10b)=1, which would force 7a+3b=$\frac{1}{3}$, impossible if a and b are integers.

(6-18) Suppose a,b $\in \mathbb{Z}$. Show that if $4|a^2 + b^2$ then a and b are not both odd.

Solution: Suppose that a and b are both odd. We know $a^2 + b^2 \equiv 0$ mod 4, and that a=2k+1, b=2l +1 for some integers k and l. Yet then $a^2 = 4k^2 + 4k + 1$ and $b^2 = 4l^2 + 4l + 1$ so $a^2 + b^2 = 4(k^2 + l^2 + l + k) + 2 \equiv 2$ mod 4 a contradiction.

(6-20) Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

Solution: Assume that $x = \frac{a}{b}, y = \frac{c}{d}$ are fully reduced rational solutions to the equation. Then we have $\frac{a^2}{b^2} + \frac{c^2}{d^2} = 3$. Then $(ad)^2 + (bc)^2 = 3(bd)^2 \equiv 0 \bmod 3$. Now any number squared is either congruent to 1 or 0 mod 3 (see Section 7 questions) and since they both add up to 0 mod 3 we must have that $(ad)^2 \equiv 0 \bmod 3$ and $(bc)^2 \equiv 0 \bmod 3$. Yet this means that $ad \equiv 0 \bmod 3$ and $bc \equiv 0 \bmod 3$ (again see my section 7 question). Thus $3|ad$ and $3|bc$. Now 3 is a prime number, so we must have that in each case 3 divides one of the integers themselves.
3 cannot divide both a and b (and similarly can't divide both c and d) or else it wouldn't be fully reduced. Hence we have two options:
a) $3|a$ and $3|c$. Yet then, plugging in, we get that $3|bd$ so 3 must divide either b or d, making one of the fractions not reduced, a contradiction.
b) The last option is $3|b$ and $3|d$. Yet then again after plugging in we must have that a or c is divisible by 3, another contradiction.

(6-24) The number $log_2 3$ is irrational.

Solution: Suppose $log_2 3 = \dfrac{x}{y}$ is rational. Then $3 = 2^{\frac{x}{y}}$ so $3^y = 2^x$. Now the LHS is odd and the RHS is even, so they can never be equal, a contradiction.

(7-12) There exists a positive real number x such that $x^2 < \sqrt{x}$

Solution: Since x is positive we can square both sides and not change the inequality: so we are looking for a positive x such that $x^4 < x^2$. Now $x = \frac{1}{2}$ works.

(7-26) The product of any n consecutive positive integers is divisible by $n!$

Solution: We want to show that for any positive integer k, $\Delta = \prod_{i=0}^{n-1}(k+i)$ is divisible by $n!$. Note that $\binom{k+n}{n} = \frac{(k+n)!}{n!k!}$ is an integer. Yet $\dfrac{\Delta}{n!} = \binom{k+n}{n}$ showing that $n!$ divides Δ.