Homework 8 Answers
David Rubinstein - Math 100 - Fall 2019

9-12) If a,b,c $\in \mathbb{N}$ and ab, ac, bc all have same parity then a, b, c all have same parity.

Solution: This is false. Take a=b=2 and c=3.

9-14) If A,B are sets then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

Solution: Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \subseteq A$ and $X \subseteq B$, so $X \subseteq A \cap B$, hence $X \in \mathcal{P}(A \cap B)$.
Similarly, let $Y \in \mathcal{P}(A \cap B)$. Then $Y \subseteq A \cap B$, so in particular, $Y \subseteq A$ and $Y \subseteq B$, hence $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

9-24) The inequality $2^x \geq x + 1$ for all positive numbers x.

Solution: Let $x = \frac{1}{2}$. Then $2^x = \sqrt{2} < 3/2$ so false.

(9-34) If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$

Solution: This is false. Let $A = \{1, 2, 3\}, B = \{3, 4, 5\}$ and let $X = \{2, 3, 4\}$

(10-4) If $n \in \mathbb{N}$ then $2 + 2 \times 3 + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

Solution: We prove this by induction. The base case, n=1 says that 2=2 which is true. Thus assume this is true for some $n \geq 1$. Then we get

$$2 + 2 \times 3 + \cdots + n(n + 1) + (n + 1)(n + 2) = \frac{n(n + 1)(n + 2)}{3} + (n + 1)(n + 2)$$

. Now we have that

$$\frac{n(n + 1)(n + 2)}{3} + (n+1)(n+2) = \frac{n(n + 1)(n + 2) + 3(n + 1)(n + 2)}{3} = \frac{(n + 1)(n + 2)(n + 3)}{3}$$

which is exactly what we wanted.

(10-8) If $n \in \mathbb{N}$ then $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Solution: We again prove this by induction. The base case, n=1 is true because $\frac{1}{2} = 1 - \frac{1}{2}$. Thus we assume this is true for some $n \geq 1$ and show this implies it is true for n+1. To that end, we have

$$\frac{1}{2!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 + \frac{-(n+2)+n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

.

Thus this is true for all natural numbers n.

(10-18) Suppose $A_1, A_2, \ldots A_n$ are all subsets of some universal set U. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$

Solution: We again prove this by induction. We first prove that $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$. We have $\overline{A_1 \cup A_2} = \{x : x \in U, x \notin A_1 \cup A_2\} = \{x : x \in U, x \notin A_!, x \notin A_2\} = \overline{A_1} \cap \overline{A_2}$ so the base case holds. Now assume this is true for $n \geq 2$. Let $B = A_1 \cup \cdots \cup A_n$. Then $\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} = \overline{B \cup A_{n+1}}$. Yet we just showed that $\overline{B \cup A_{n+1}} = \overline{B} \cap \overline{A_{n=1}}$. Then by our inductive hypothesis, we have that $\overline{B} = \overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$ so combining this we get that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} = \overline{A_1} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}$ and we are done.

(10-22) If $n \in \mathbb{N}$ prove that $(1 - \frac{1}{2})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}$

Solution: For n=1, this says $\frac{1}{2} \geq \frac{1}{4} + \frac{1}{4}$ which is true, so the base case holds. Now assume it is true for some $n \geq 1$. Then

$$
\begin{aligned}
(1-\frac{1}{2})(1-\frac{1}{4})\cdots(1-\frac{1}{2^n})(1-\frac{1}{2^{n+1}}) &\geq (\frac{1}{4}+\frac{1}{2^{n+1}})(1-\frac{1}{2^{n+1}}) && (1)\\
&= \frac{1}{4} - \frac{1}{4(2^{n+1})} + \frac{1}{2^{n+1}} - \frac{1}{(2^{n+1})^2} && (2)\\
&= \frac{1}{4} + \frac{1}{2^{n+1}}(-\frac{1}{4} + 1 - \frac{1}{2^{n+1}}) && (3)\\
&= \frac{1}{4} + \frac{1}{2^{n+1}}(\frac{3}{4} - \frac{1}{2^{n+1}}) && (4)\\
&\geq \frac{1}{4} + \frac{1}{2^{n+1}}(\frac{3}{4} - \frac{1}{4}) && (5)\\
&= \frac{1}{4} + \frac{1}{2^{n+1}}(\frac{1}{2}) && (6)\\
&= \frac{1}{4} + \frac{1}{2^{n+2}} && (7)
\end{aligned}
$$

where the inequality between (4) and (5) holds because, for $n \geq 1$, $\frac{1}{2^{n+1}} \leq \frac{1}{4}$, so when we subtract off the terms the inequality flips.