# Functions and Cardinality

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Question 1

a) Come up with a function that is injective but not surjective.
b) Come up with a function that is surjective but not injective.
c) Come up with a function that is both injective and surjective.

Question 2

a) Let $f : A \to B$ be a function. Recall that we showed in section that for $X \subseteq A$ it is not always true that $f^{-1}(f(X)) = X$. However, prove that for any subset $X \subseteq A, X \subseteq f^{-1}(f(X))$
b) Now consider a subset $Y \subseteq B$. State and prove an analogous version of part a regarding the relationship of Y to $f(f^{-1}(Y))$. It might help to draw some silly pictures here to convince yourself of what the conjecture should be.
c) Again, let $f : A \to B$ be a function. Show that f is injective iff for any $X \subseteq A, f^{-1}(f(X)) = X$
d) State and prove the analogous claim to part c for a function being surjective. That is, give necessary and sufficient criteria for a function f to be surjective based on its the relationship of $Y \subseteq B$ to $f(f^{-1}(Y))$.

Question 3- Collections of Functions as a Group

Recall that if $f : A \to B, g : B \to C$ were functions we could form the composite function $g \circ f : A \to C$ but not the composite function $f \circ g$. However, if our domain and codomain are the same, we can do both $g \circ f$ and $f \circ g$ for any two functions $f, g : A \to A$. Finally, recall that we showed

(or at least claimed) that function composition is associate.
Now, we consider the set $Funct(A) = \{f : f : A \to A \text{ is a function }\}$. Inside this set we have $Invert(A) = \{g : g : A \to A \text{ is bijective }\}$. This set Invert(A) shares some very similar properties to $\mathbb{Z}_n$ that I spelled out in Section 7 questions. Namely, I claimed that the addition of equivalence classes we defined for $\mathbb{Z}_n$ turned it into a "group".
What we have basically just showed is that, if we define addition on the set Invert(A) as function composition, then we get that Invert(A) is a group!! This provides maybe one of the more compelling reasons for why we should care about group theory: Given any set (!!), we can form the group of bijective functions on that set! This shows that groups arise extremely naturally everywhere we look in mathematics.