Determinants

August 9, 2021

Question 1

Determine if the following matrices are invertible. (No need to find the inverse)

a)A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

b) A= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

c) A= $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 0 & 3 \end{pmatrix}$.

Question 2

Compute the determinants of the following matrices using any method you want.

a) A= $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$.

b) B= $\begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 3 & 4 & 6 \\ 1 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

c) C= $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 6 & 9 & -26 \\ 0 & 0 & 8 & 12 & 18 \\ 2 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$.

Question 3

Say whether the following are true or false. If false explain why or give a counter example.

a) det(A+B)= det(A) + det(B) for any $n \times n$ matrices A,B.

b) Suppose A and B are two 36×36 matrices with det(A)=1063 and det(B)=2. Then the matrix AB is invertible.

c) Suppose A is a 3×3 matrix with det(A)=12. There exists some vector $\vec{b} \in \mathbb{R}^3$ such that there is no $\vec{x} \in \mathbb{R}^3$ with $A\vec{x} = \vec{b}$.

d) Suppose A is a $10,340 \times 10,340$ matrix with det(A)=0. Then there is some nonzero vector $\vec{x} \in \mathbb{R}^{10,340}$ such that $A\vec{x} = \vec{0}$

Challenge Question for your enjoyment

Consider the matrix A= $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$. This matrix has det(A)=2=(4-3)(4-2)(3-2) where I've expressed 2 in this weird way as a hint.

Now consider the matrix B= $\begin{pmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{pmatrix}$. This has det(B)=12=(5-

4)(5-3)(5-2)(4-2)(4-3) again written in this weird way as a hint.

Without plugging into a calculator, find the determinant of the matrix

$$C = \begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1,296 \end{pmatrix}.$$

Such matrices are called "Vandermonde Matrices" and they actually turn up in mathematics. For example, in so called "algebraic number theory" (more specifically in finite number field extensions) in "Galois theory" and in "Group Representation Theory" this matrix is used in proving some key results. Furthermore, it is also used in Error correcting codes, and in computing Discrete Fourier Transforms, with applications to music.