Intro to Abstract Vector Spaces

August 13, 2021

Question 1

Show that $\mathbb{R}_4$[x] is a vector space. That is, show that the addition and scalar multiplication we defined satisfies all the properties of being a vector space.

Question 2

a) Is 2+4x $\in$ span $\left(1 + x, 1 - 3x\right)$

b) Is $2 + 8x + 11x^2 \in$ span $\left(1 + 4x, 1 + 8x + 6x^2, -1 - 12x - x^2\right)$

c) Is $1 - x - 8x^2 \in$ span $\left(1, 1 + x + 4x^2, -x - 4x^2\right)$

Question 3

For the following, give an example if one exists, or state it is not possible. If it is not possible, explain why.

a) A sequence of 4 vectors that span $M_{2\times3}(\mathbb{R})$

b) A sequence of 3 Linearly Independent vectors in $\mathbb{R}_3[x]$.

c) A sequence of 3 Linearly independent vectors in $\mathbb{R}_2[x]$ that are not a basis.

d) A sequence of 9 spanning vectors in $M_{3\times3}(\mathbb{R})$ that are not a basis.

Question 4

Consider the following two bases for $\mathbb{R}_2[x]$:

$$\mathcal{B} = \{1, x, x^2\}$$
$$\mathcal{C} = \{2 + x, 3 + x, x - x^2\}$$

a) Find $\mathcal{P}_{\mathcal{B}\to\mathcal{C}}$ (the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$)
b) Find $[8 - 12x + 36x^2]_{\mathcal{B}}$
c) Find $[8 - 12x + 36x^2]_{\mathcal{C}}$

Question 5
Consider the following two bases for $M_{2\times 2}(\mathbb{R})$:

$$\mathcal{B} = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$
$$\mathcal{C} = \{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}$$

a) Suppose m is a matrix and $[m]_{\mathcal{C}} = \begin{pmatrix} 3 \\ 8 \\ 12 \\ -2 \end{pmatrix}$. Find what the matrix m is.

b) Find $[m]_{\mathcal{B}}$
c) Find $\mathcal{P}_{\mathcal{C}\to\mathcal{B}}$ and confirm that $[m]_{\mathcal{B}} = \mathcal{P}_{\mathcal{C}\to\mathcal{B}}[m]_{\mathcal{C}}$