Linear Transformations and Rank-Nullity

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Question 1

For the following three Linear Transformations: 1) find a basis for the range; 2) find the kernal; and 3) verify the rank-nullity theorem in each case.

a) $T : \mathbb{R}_3[x] \to M_{2\times 2}(\mathbb{R})$ given by

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a & b-c \\ d & a+c \end{pmatrix}$$

b) $T : M_{3\times 2}(\mathbb{R}) \to \mathbb{R}^3$ given by

$$T(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}) = \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \\ a_{31} + a_{32} \end{pmatrix}$$

c) $T : \mathbb{R}^4 \to \mathbb{R}_3[x]$ by

$$T(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) = (a - c) + (b - c)x + (a - b)x^3$$

Question 2

Write the matrix associated to the 3 linear transformations above with respect to the following given bases:

a)

$$\mathcal{B}_{\mathbb{R}_3[x]} = (1, x, x^2, x^3)$$
$$\mathcal{C}_{M_{2\times 2}(\mathbb{R})} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

b)

$$\mathcal{B}_{M_{3\times 2}(\mathbb{R})} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$
$$\mathcal{C}_{\mathbb{R}^3} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

c)

$$\mathcal{B}_{\mathbb{R}^4} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$
$$\mathcal{C}_{\mathbb{R}_3[x]} = \left(1, x, x^2, x^3 \right)$$

Question 3

For the following, give an example if one exists, or state it is not possible. If it is not possible, explain why.

a) An injective linear transformation $T : M_{3\times 3}(\mathbb{R}) \to \mathbb{R}_6[x]$
b) A surjective linear transformation $T : \mathbb{R}_3[x] \to M_{2\times 2}(\mathbb{R})$
c) An injective linear transformation between two vectors spaces of the same dimension that is not an isomorphism.
d) A surjective linear transformation $T : \mathbb{R}_5[x] \to \mathbb{R}^3$ with nullity(T)$= 4$
e) An injective linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ with rank(T)=3

Question 4

State whether the following are true or false: if they are false give a counter example.

a) Every linear transformation $T : V \to W$ with $dim(V) < dim(W)$ is injective.
b) Every surjective linear transformation $T : V \to W$ with $dim(V) = dim(W)$ is an isomorphism.
c) Every linear transformation $T : V \to W$ with $dim(V) > dim(W)$ is surjective.
d) Suppose $V \cong W$ (V is isomorphic to W). Then every linear transformation $T : V \to W$ is an isomorphism.